Buckling Plasticity Factors

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Column & Plate Buckling Equations

Column

$$F_{cr} = \frac{\eta \pi^2 E}{\left(L'/\rho\right)^2}$$

where: $L' = \frac{L}{\sqrt{c}}$ $\rho^2 = \frac{I}{A}$

Plate

$$F_{cr} = \frac{\eta k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2$$ **or** $$F_{cr} = \frac{\eta E}{(b/t)_e^2}$$

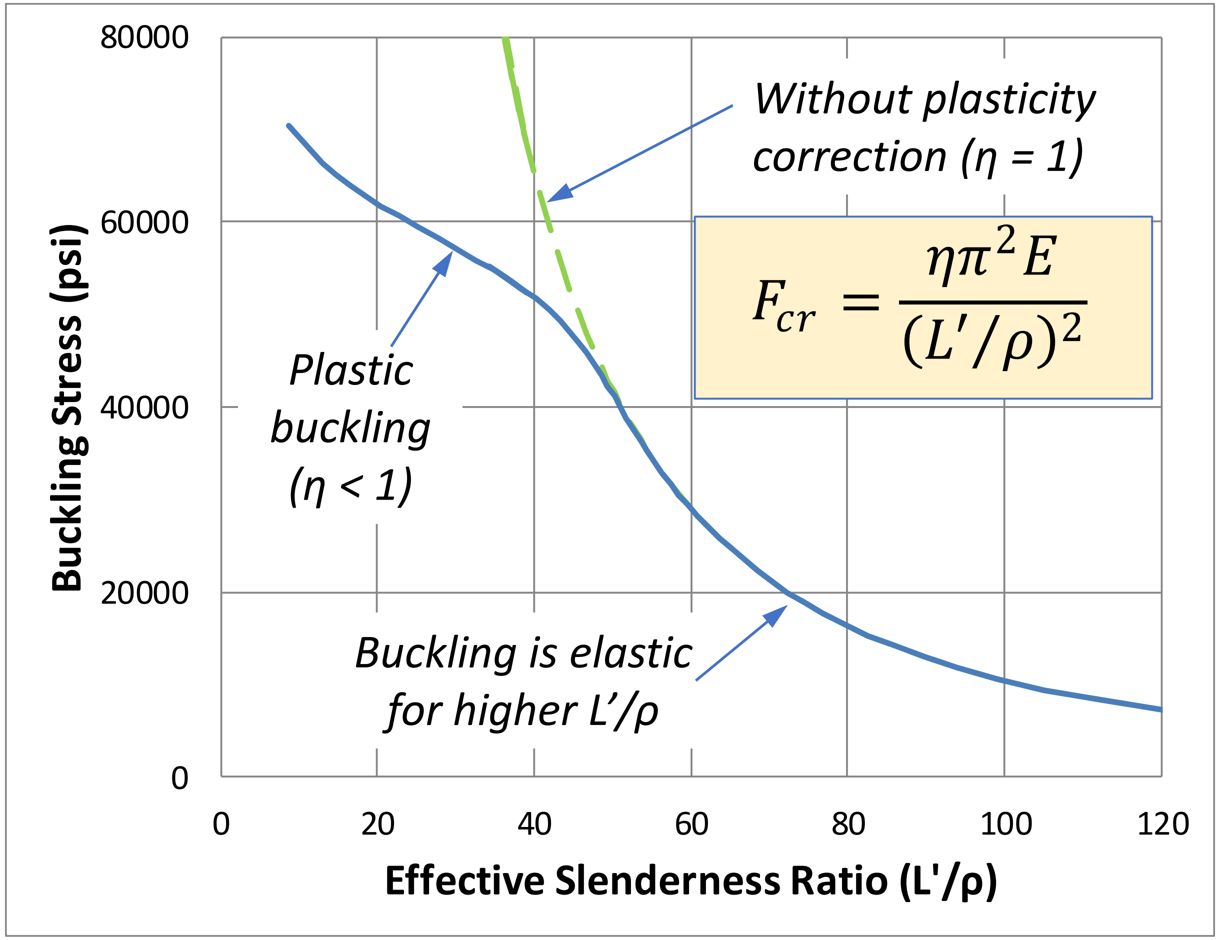
where: $(b/t)_e = \frac{(b/t)}{\sqrt{K}}$

$$K = \frac{k\pi^2}{12(1-\nu^2)}$$

η = Plasticity Correction Factor (number ≤ 1)
Not same for columns & plates (see later)
Reduces to elastic buckling when η = 1

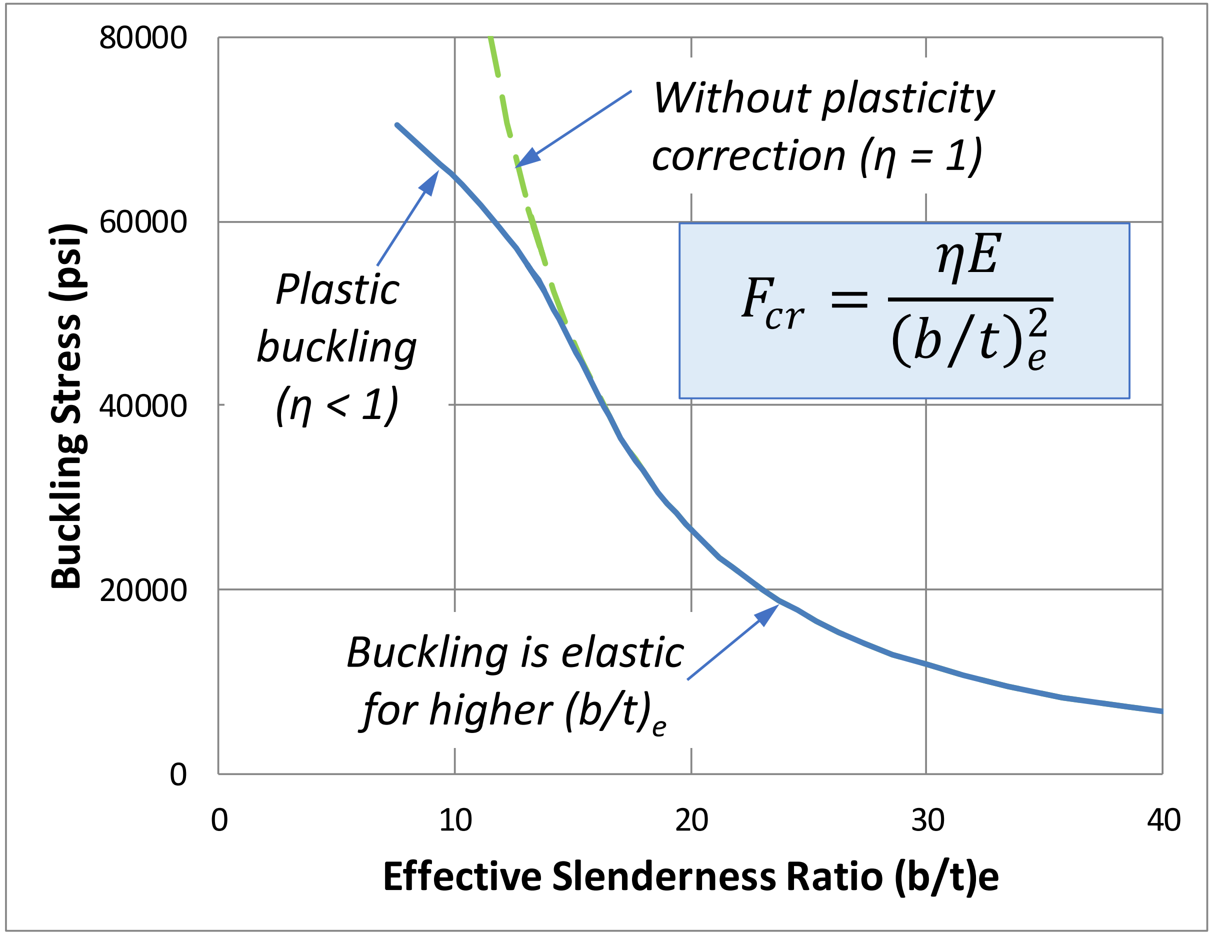
Plasticity Reduces Buckling Stresses ($\eta \leq 1$)

Column Buckling



$$F_{cr} = \frac{\eta \pi^2 E}{(L'/\rho)^2}$$

Plate Buckling

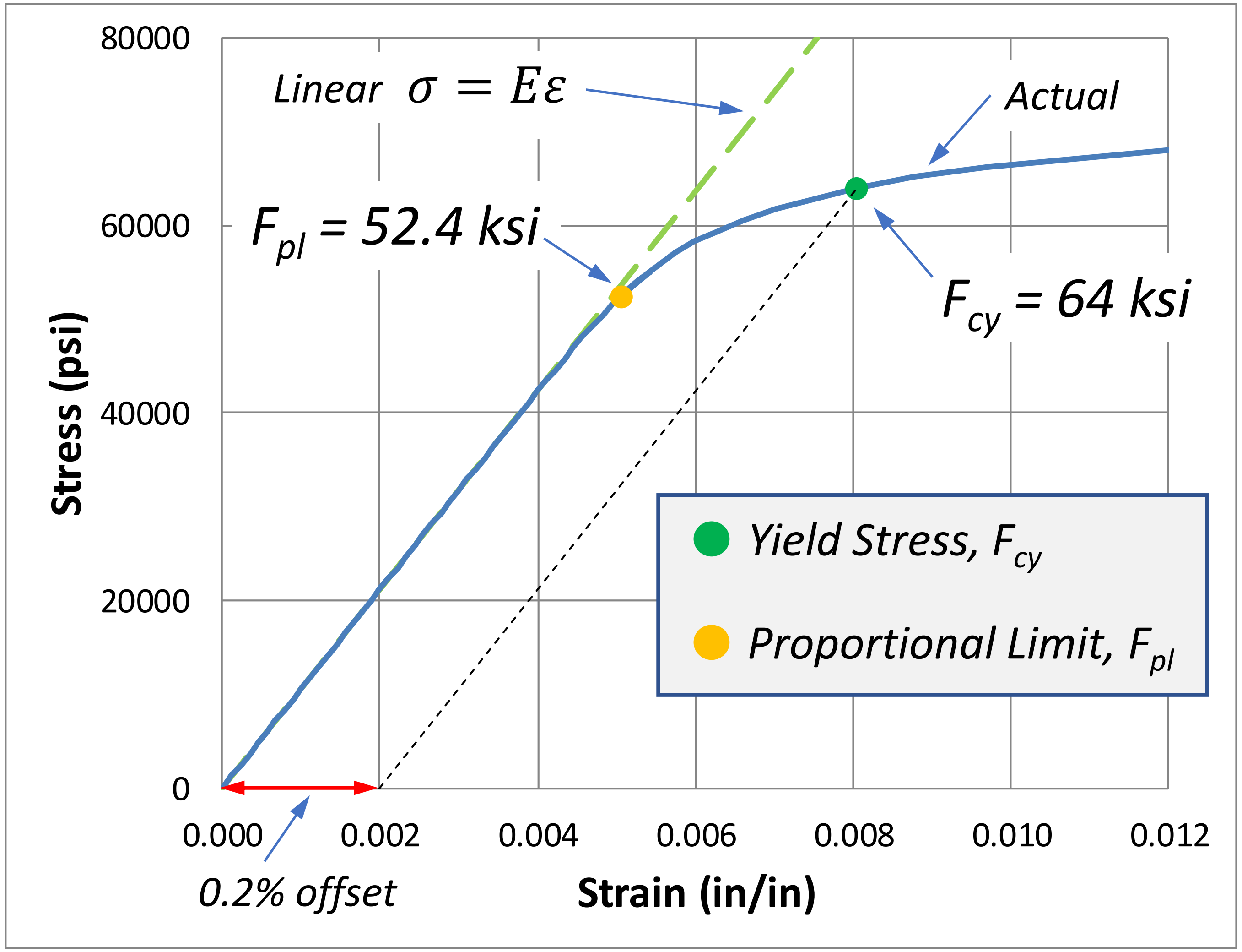


$$F_{cr} = \frac{\eta E}{(b/t)_e^2}$$

η = Plasticity Correction Factor

Compressive Stress-Strain Curve

Typical Aluminum Alloy



Note that the stress-strain curve deviates from linearity <u>below</u> the yield stress.

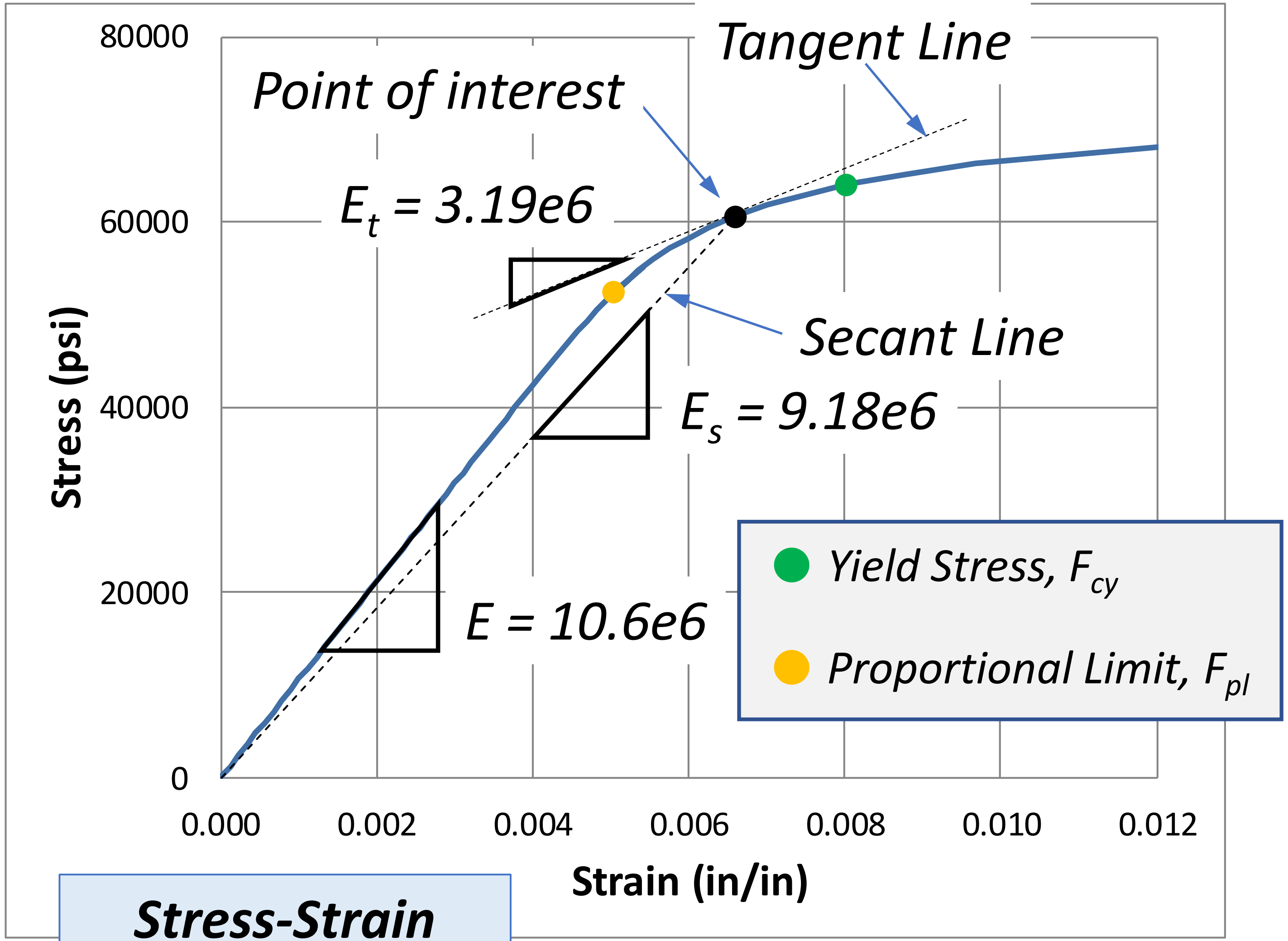
<u>Yield Stress F_{cy}</u>

Stress where deviation from initial straight line is 0.002 in/in (0.2% offset). This offset is shown as the dotted line in this plot.

<u>Proportional Limit F_{pl}</u>

Stress where deviation from initial straight line begins. Since it can be hard to determine exactly where this occurs, it is often defined as a deviation of 0.0001 in/in (0.01% offset).

Plasticity Factors are functions of E_t and E_s



Stress-Strain

Besides the initial Young's Modulus *E*, two other moduli are used when analyzing plasticity effects:

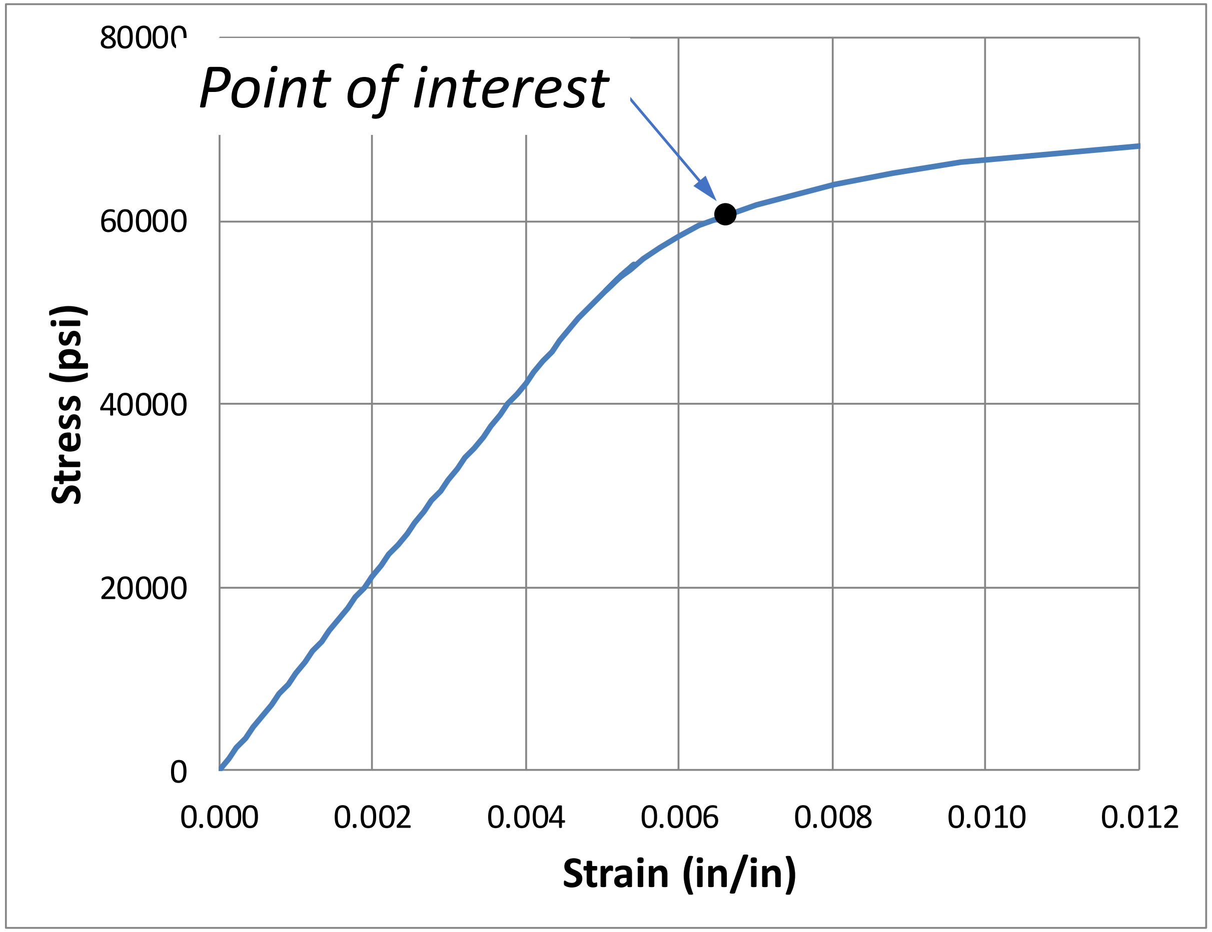
E_t = Tangent Modulus (psi)
E_s = Secant Modulus (psi)

The values of E_t and E_s at the point of interest are as shown. Note how they compare to the initial Modulus.

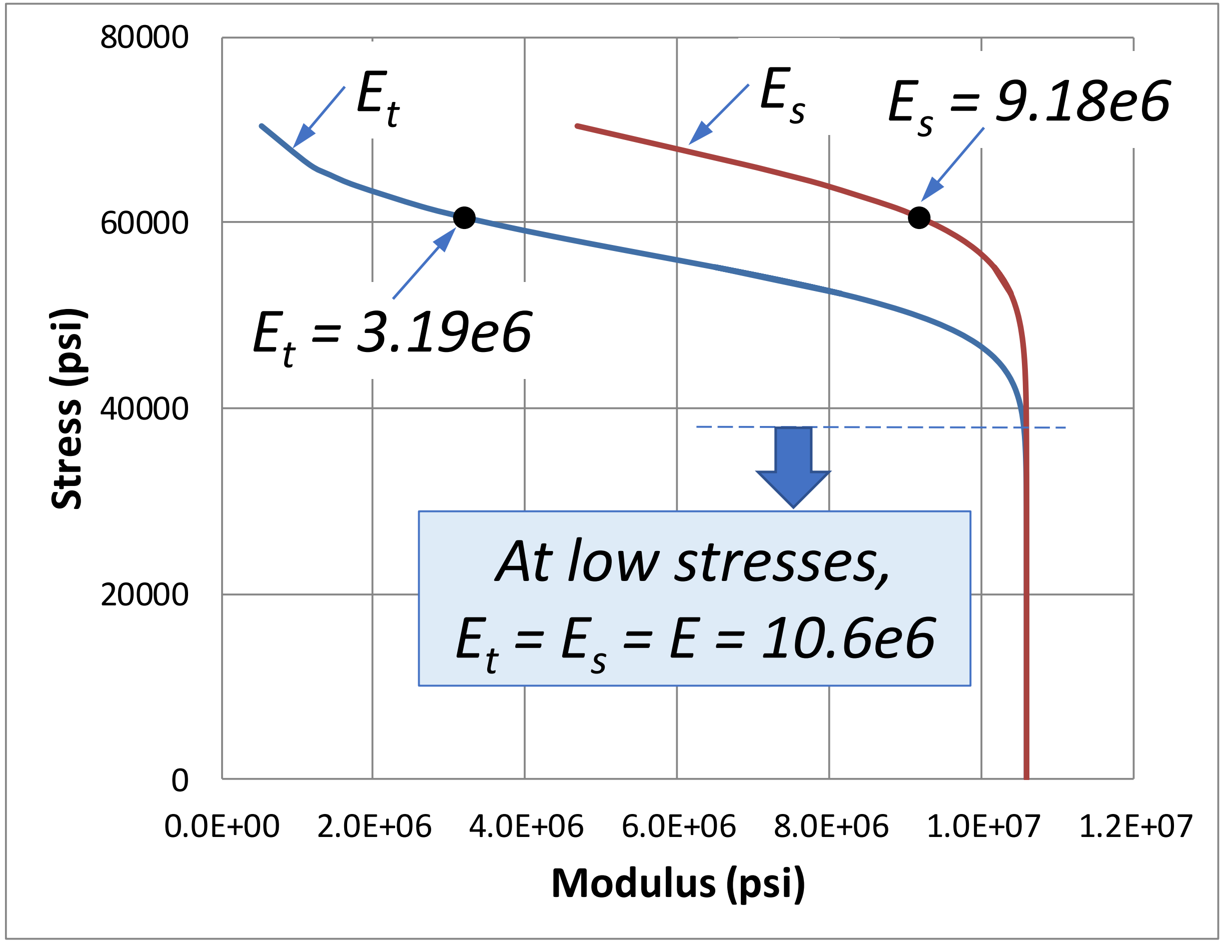
E_t and E_s vary along the curve. They are functions of the stress, they are not constants!

Tangent & Secant Moduli vary with Stress

Stress-Strain



Tangent & Secant Moduli as functions of stress



Ramberg-Osgood Relation

Ramberg-Osgood Relation

- While actual stress-strain curves are obtained by testing, it is useful to have a mathematical expression to represent the curve for analysis work
- Although several expressions have been proposed, the Ramberg-Osgood representation has been the most commonly used for Aerospace Structural Analysis
- There are a few variations of the basic Ramberg-Osgood formulation that will be shown below
- For analysis of buckling in the plastic range, the main use of the Ramberg-Osgood relation is to obtain mathematical expressions for the Tangent and Secant Moduli, which are needed to compute Plasticity Correction Factors

Basic Ramberg-Osgood Stress-Strain Relation

The total strain is broken into elastic and plastic parts

$$\varepsilon = \varepsilon_e + \varepsilon_p$$

The elastic strain is linear with stress

$$\varepsilon_e = \frac{\sigma}{E}$$

Elastic strain dominates at low stresses

The plastic strain is a power law with stress

$$\varepsilon_p = K\left(\frac{\sigma}{E}\right)^n$$

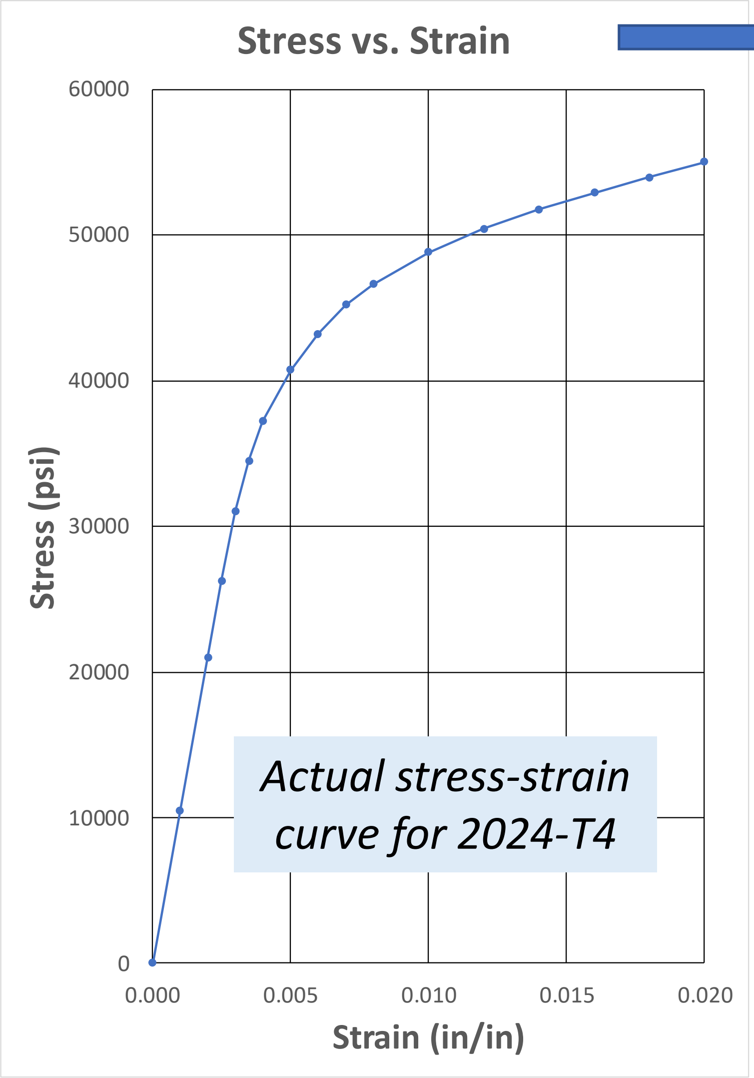
Plastic strain dominates at high stresses

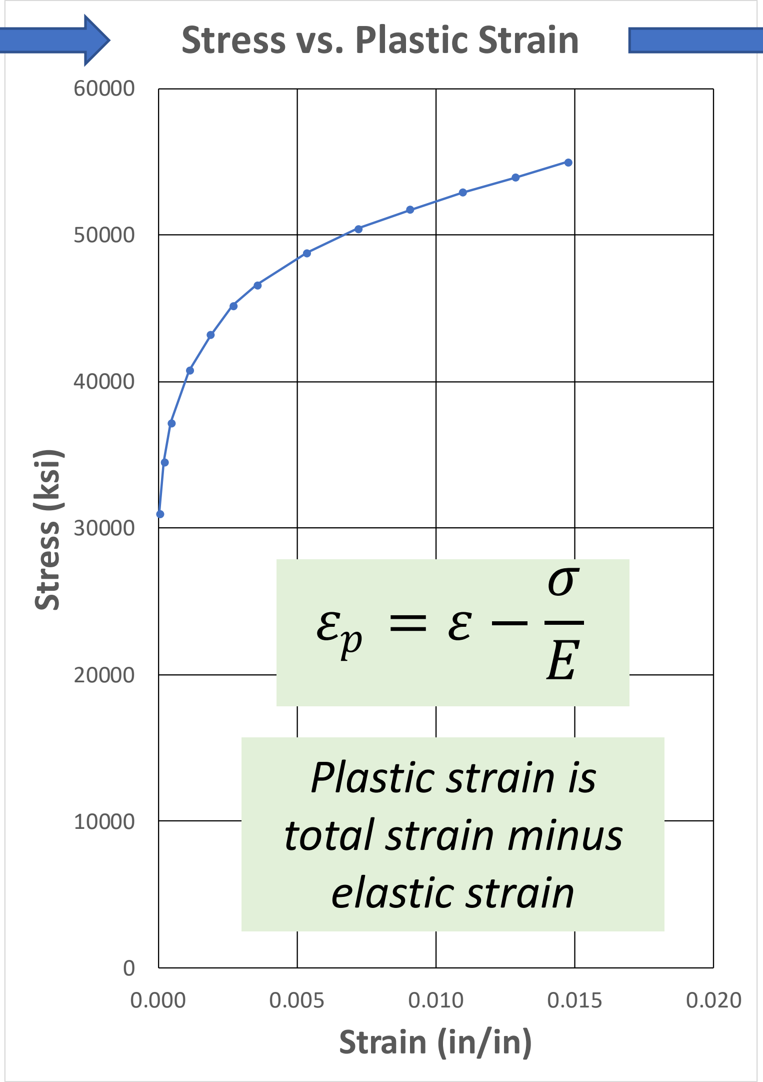
E, K, n are constants for a specific material

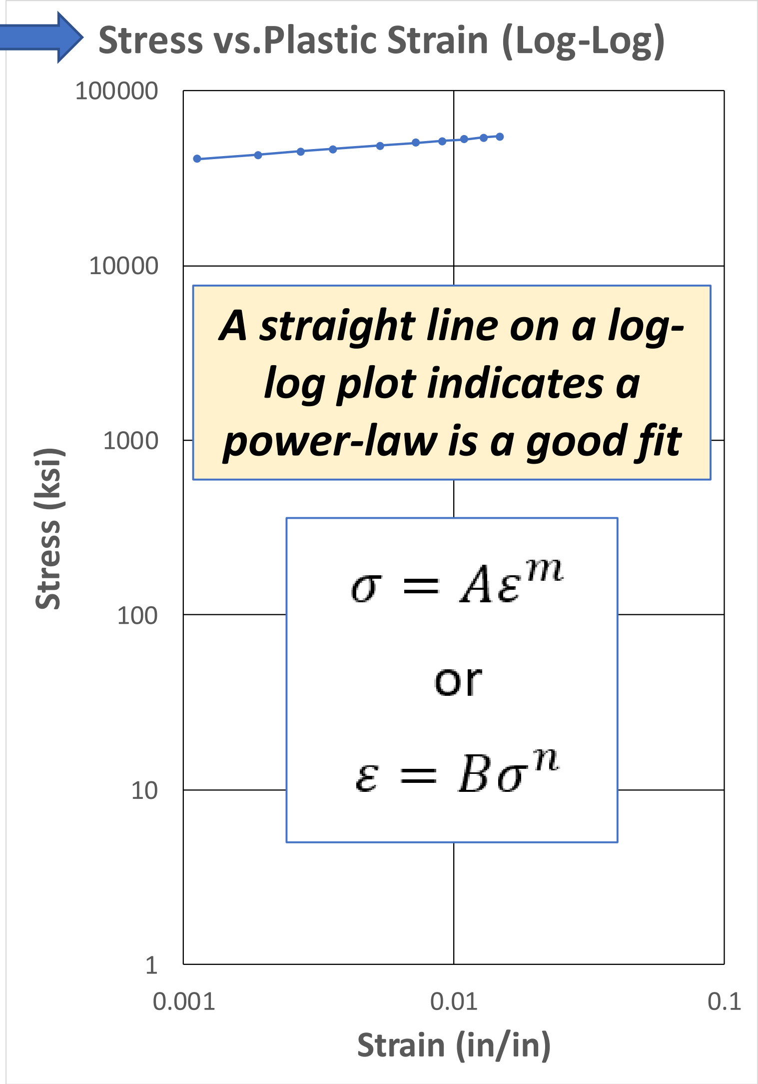
$$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$$

The resulting stress-strain relation is nonlinear; "n" = Ramberg-Osgood shape parameter

Why Power Law for Plastic Strain?

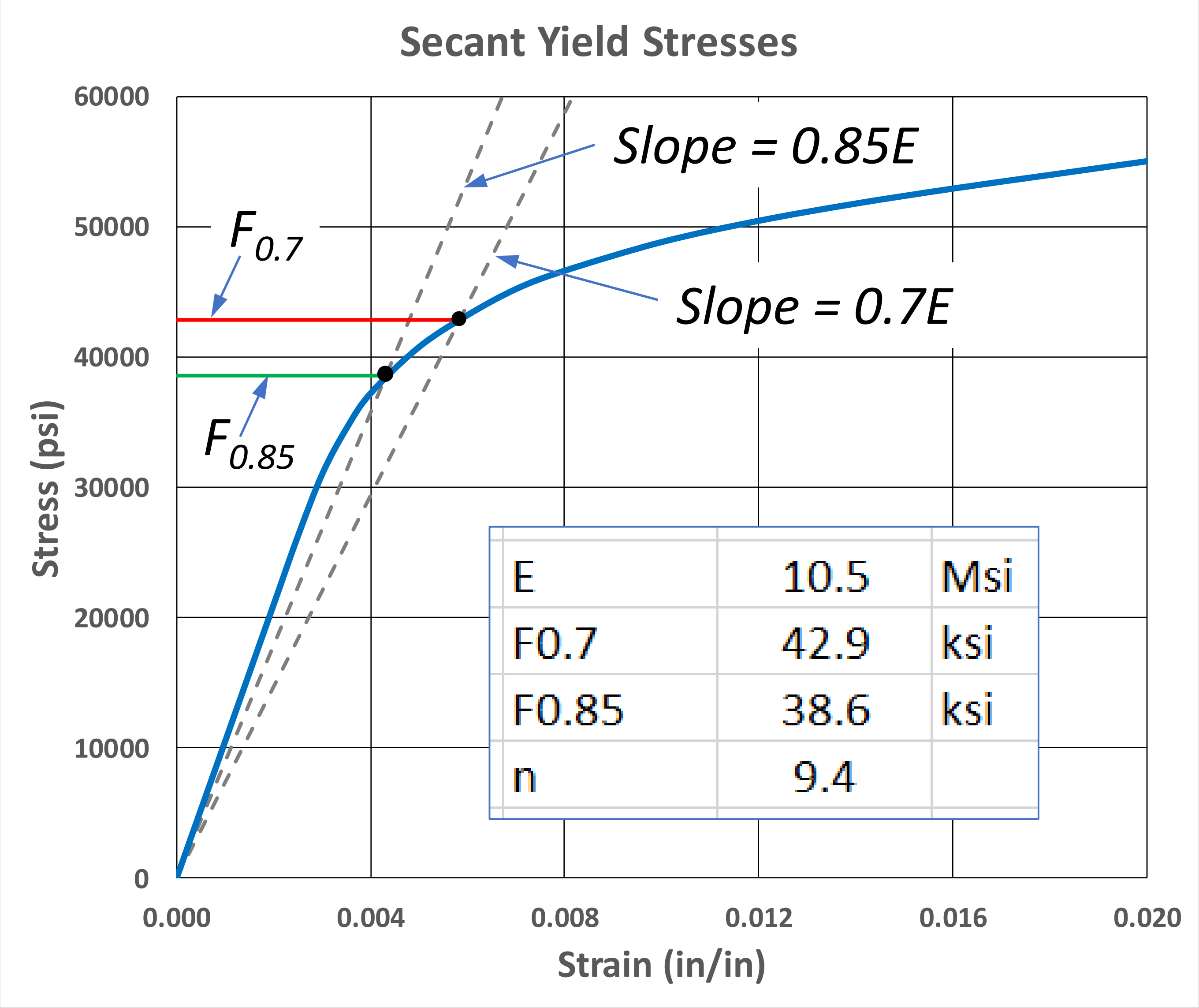






Variation 1: Original Ramberg-Osgood

NACA-TN-902



Fit curve through 2 Secant Yield Stresses to obtain exponent "n"

$$n = 1 + \frac{\log\left(\frac{17}{7}\right)}{\log\left(\frac{F_{0.7}}{F_{0.85}}\right)}$$

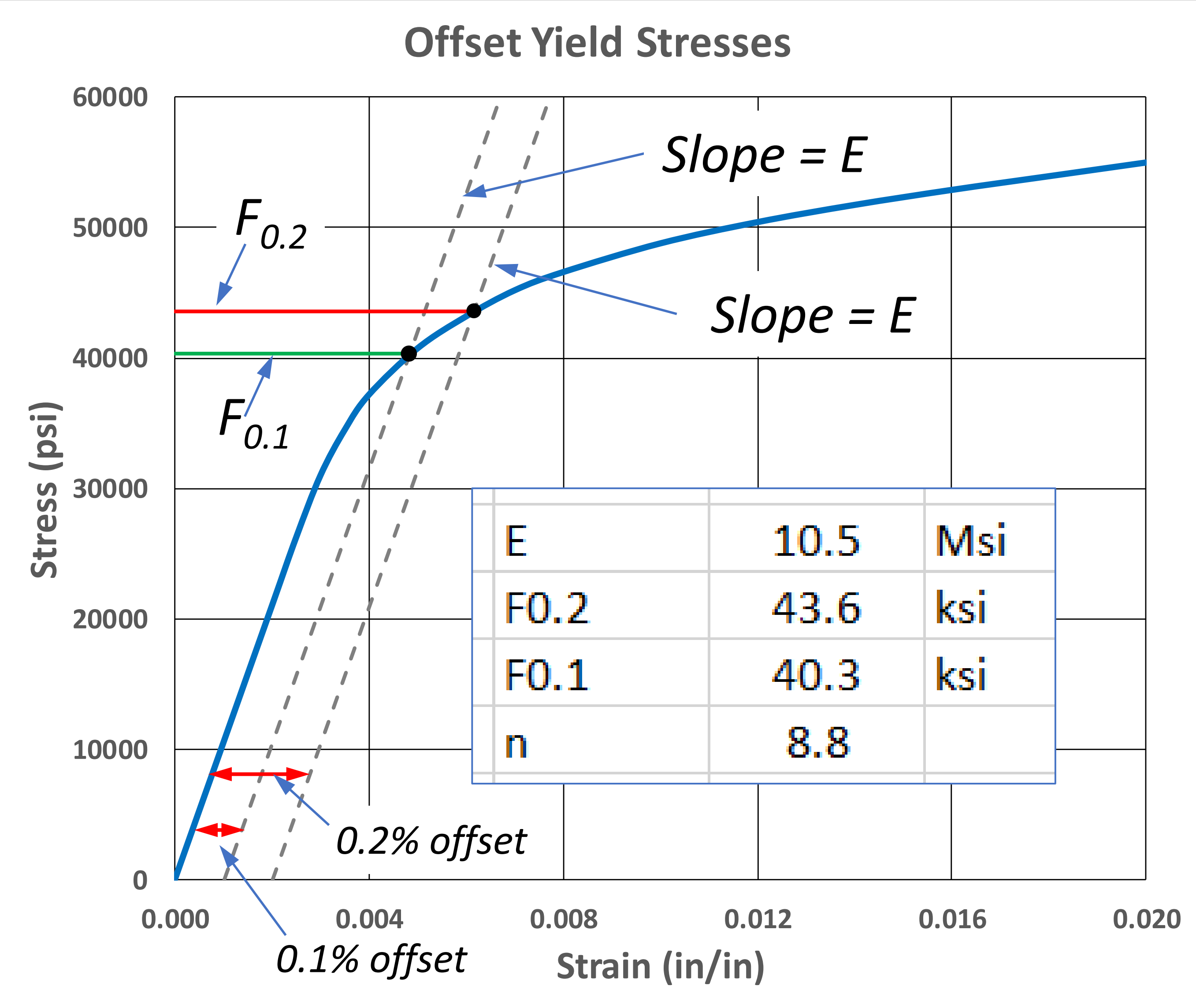
$F_{0.7}$ = Stress at secant slope of 0.7*E* (psi)
$F_{0.85}$ = Stress at secant slope of 0.85*E* (psi)

$$\varepsilon = \frac{\sigma}{E} + \frac{3}{7}\frac{F_{0.7}}{E}\left(\frac{\sigma}{F_{0.7}}\right)^n$$

Stress-strain curve using 3 parameters: E, n, $F_{0.7}$

Variation 2: Hill

NACA-TN-927



E	10.5	Msi
F0.2	43.6	ksi
F0.1	40.3	ksi
n	8.8	

Fit curve through 2 Offset Yield Stresses to obtain exponent "n"

$$n = \frac{\log\left(\frac{.002}{.001}\right)}{\log\left(\frac{F_{0.2}}{F_{0.1}}\right)}$$

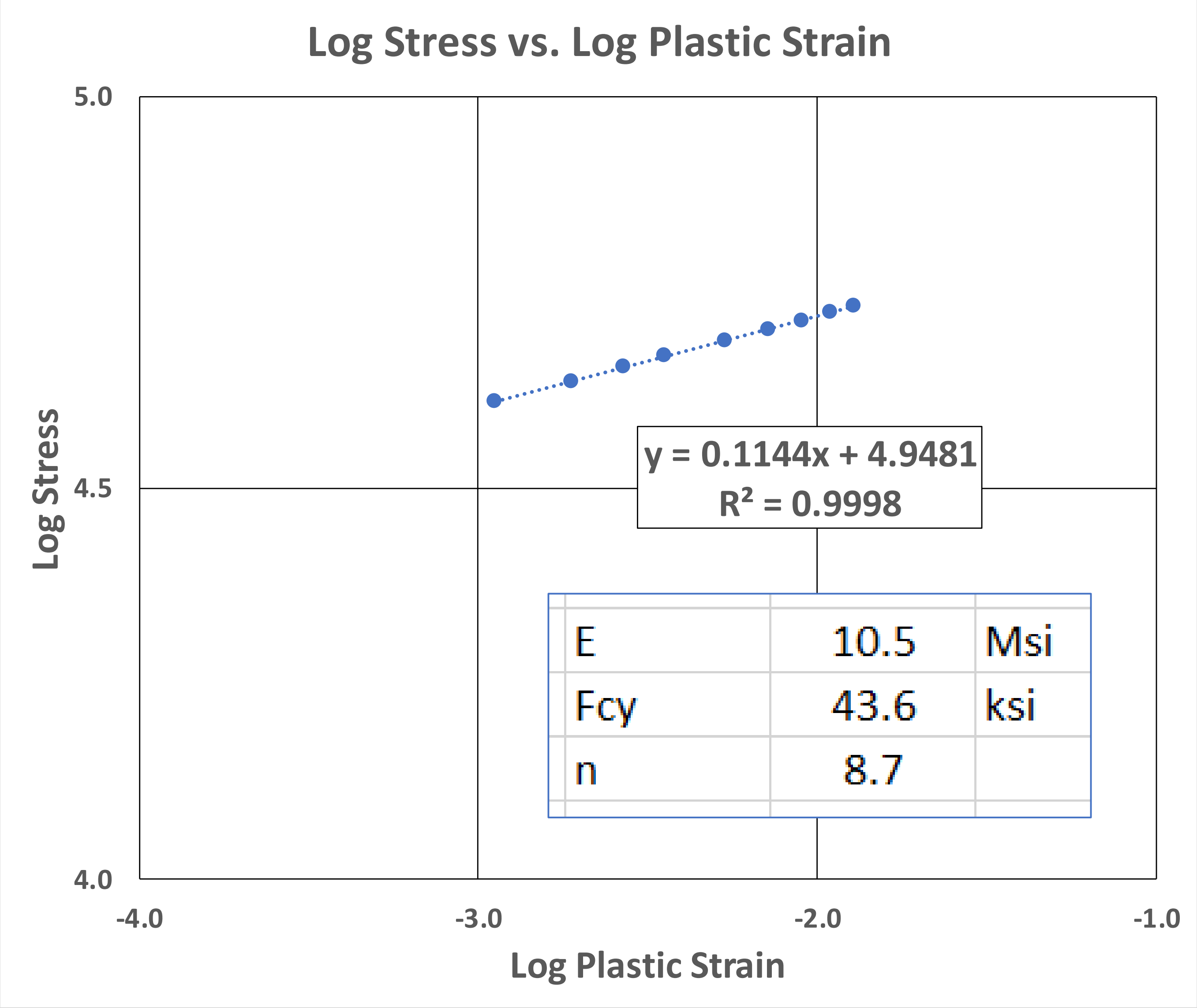
$F_{0.1}$ = Stress at offset of 0.1% (psi)
$F_{0.2}$ = Stress at offset of 0.2% (psi)

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{F_{0.2}}\right)^n$$

Stress-strain curve using 3 parameters: E, n, $F_{0.2}$

Variation 3: MIL-HDBK-5 Variation

MIL-HDBK-5J



"n" determined from linear fit of Log plastic strain vs. Log stress plot

m = slope = 0.1144
n = 1/m = 8.7

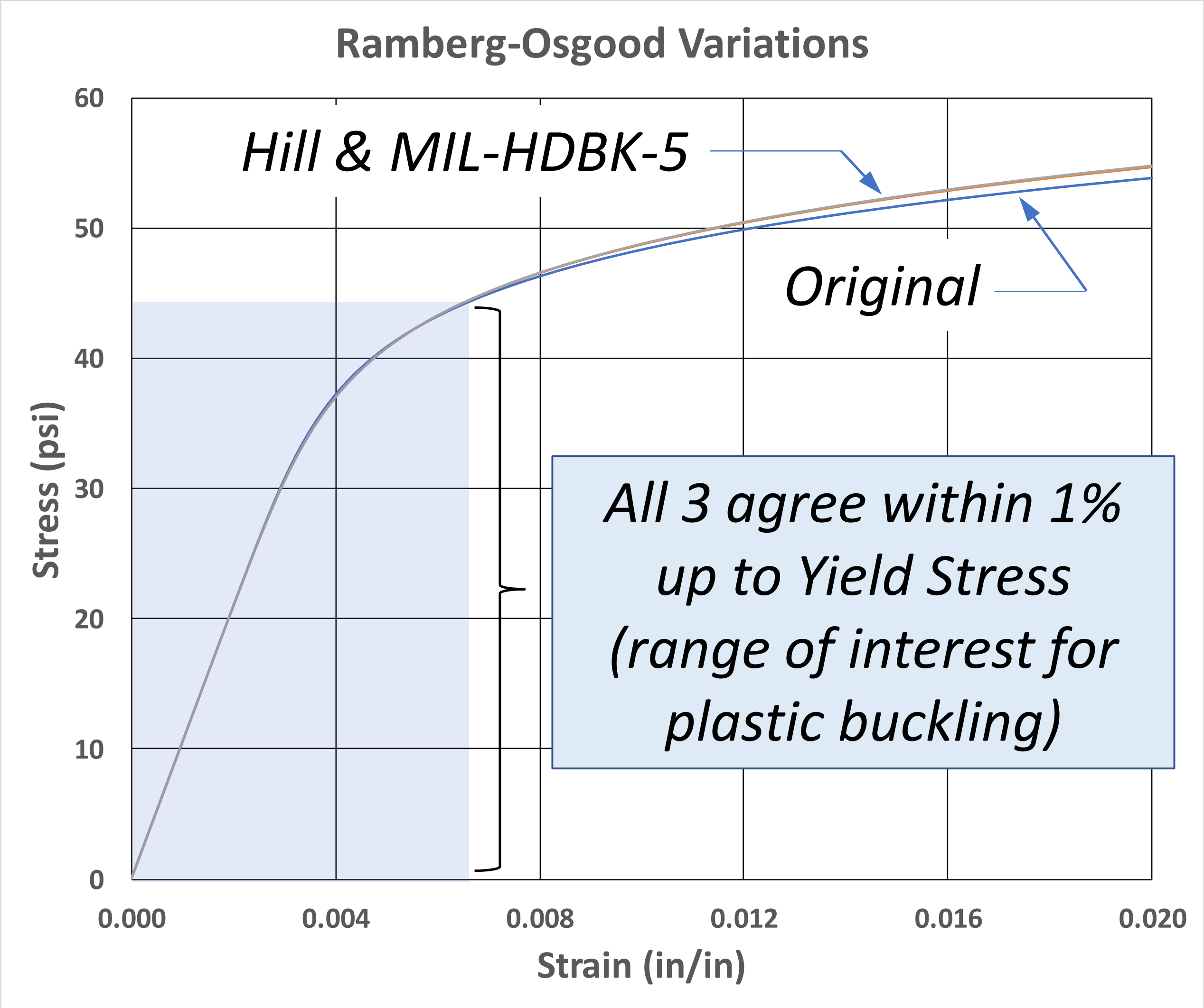
Plot log stress vs. log plastic strain over range of interest. Inverse slope of the best fit line is the desired exponent *n*.

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{F_{0.2}}\right)^n$$

Stress-strain curve using 3 parameters: E, n, $F_{0.2}$

Compare Variations

Although shape parameters vary slightly, differences in stress-strain curves for this alloy in range of interest are negligible



Original

E	10.5	Msi
F0.7	42.9	ksi
n	9.4	

$$\varepsilon = \frac{\sigma}{E} + \frac{3}{7}\frac{F_{0.7}}{E}\left(\frac{\sigma}{F_{0.7}}\right)^n$$

Hill

E	10.5	Msi
F0.2	43.6	ksi
n	8.8	

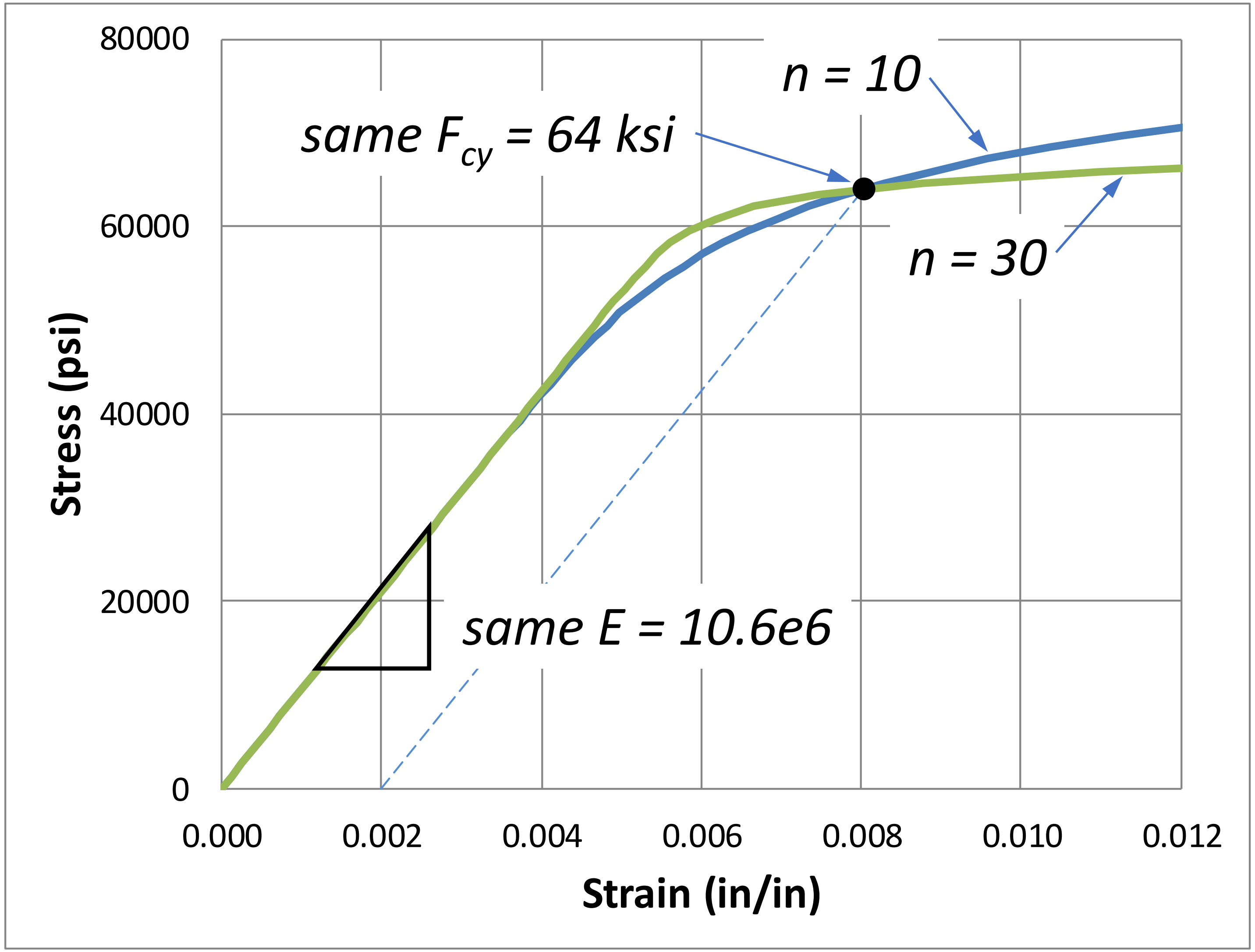
$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{F_{0.2}}\right)^n$$

MIL-HDBK-5

E	10.5	Msi
F0.2	43.6	ksi
n	8.7	

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{F_{0.2}}\right)^n$$

Effect of Shape Parameter "n"



Comparing 2 stress-strain curves with same E and Fcy, but different values of "n"

Lower values of n:
More gradual transition from elastic to plastic, plasticity effects begin at lower stresses

Higher values of n:
More abrupt transition from elastic to plastic (sharper knee)

Aside: Full-Range Tension Stress-Strain Curve

The previous curves were for compression and only covered stresses a bit past the yield stress

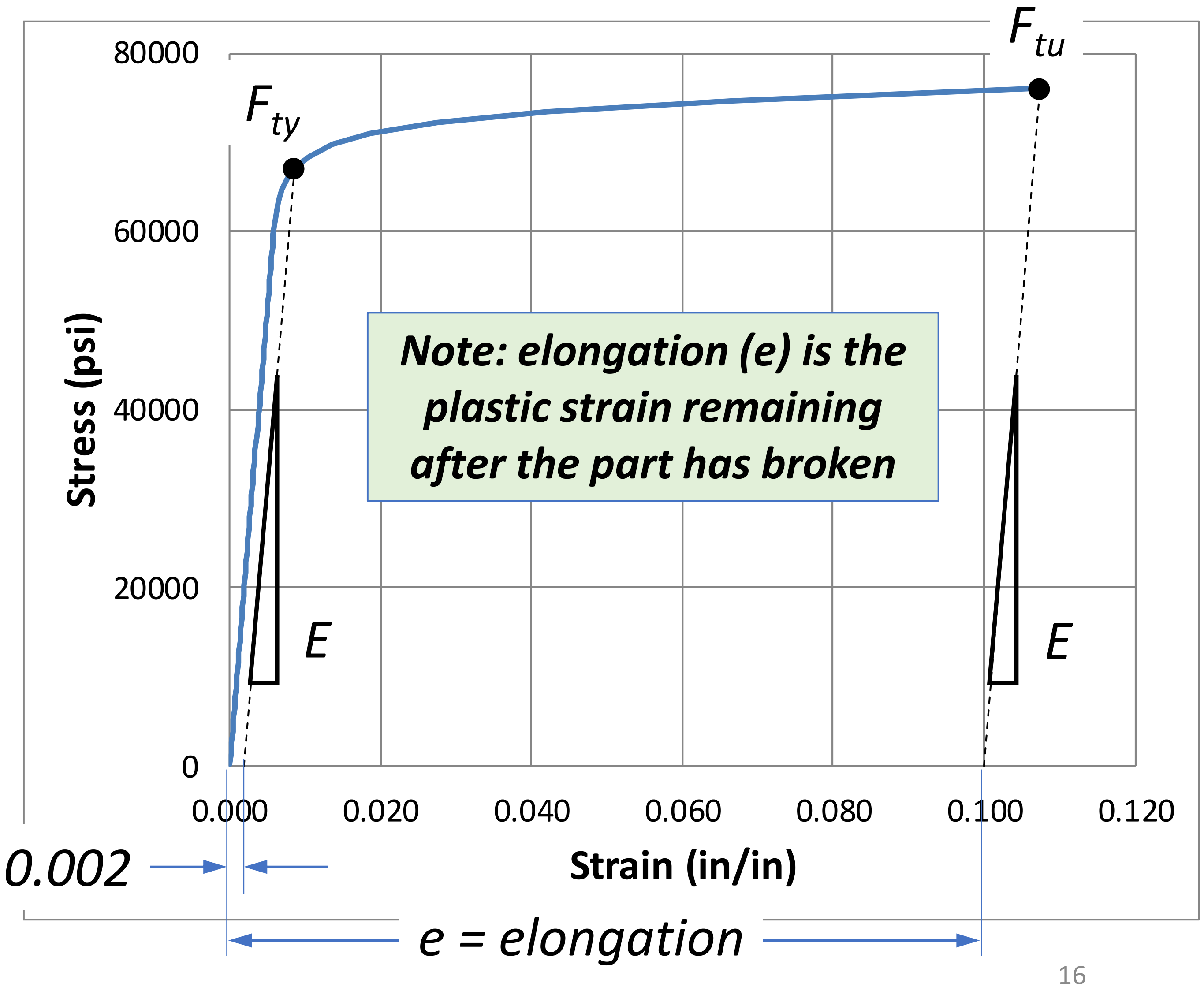
Full range tension curves are useful for plastic bending analysis.

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{F_{ty}}\right)^n$$

where:

$$n = \frac{\log\left(\frac{e}{.002}\right)}{\log\left(\frac{F_{tu}}{F_{ty}}\right)}$$

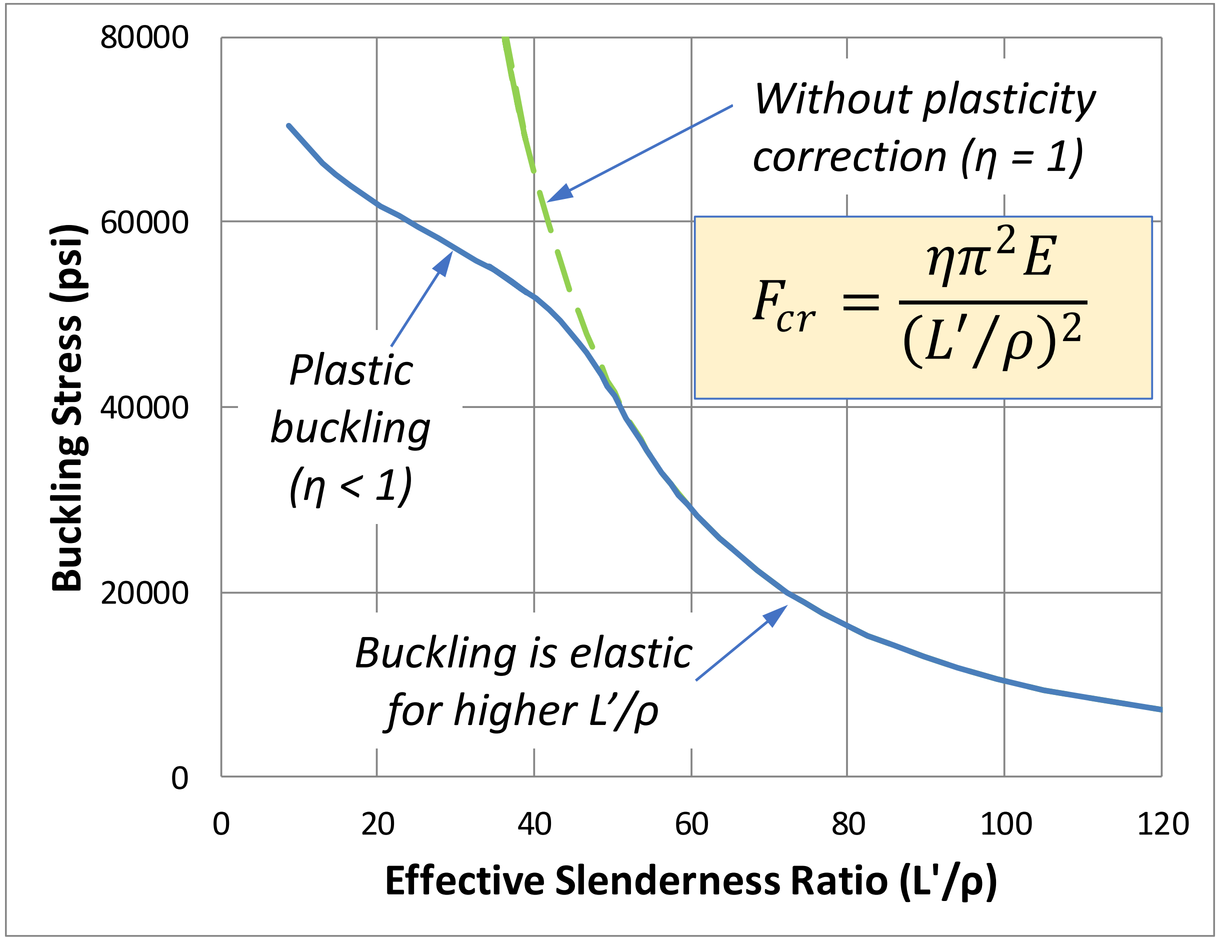
Similar to Hill formulation, except use F_{ty} and Ftu as the 2 offset stresses



Plasticity Factor for Column Buckling

Plasticity Reduction Factor for Columns

Column Buckling



$$F_{cr} = \frac{\eta\pi^2 E}{(L'/\rho)^2}$$

$$F_{cr} = \frac{\eta\pi^2 E}{\left(L'/_{\rho}\right)^2}$$

$$L' = \frac{L}{\sqrt{c}}$$

$$\rho^2 = \frac{I}{A}$$

η = Plasticity Correction Factor

where: $\eta = \frac{E_t}{E}$

Column Buckling using Ramberg-Osgood

Ramberg-Osgood

$$\varepsilon = \frac{f}{E} + \frac{3}{7}\frac{F_{0.7}}{E}\left(\frac{f}{F_{0.7}}\right)^n$$

Tangent Modulus

$$E_t = \frac{E}{1 + \frac{3}{7}n\left(\frac{f}{F_{0.7}}\right)^{n-1}}$$

Plasticity Factor

$$\eta = \frac{E_t}{E} = \frac{1}{1 + \frac{3}{7}n\left(\frac{f}{F_{0.7}}\right)^{n-1}}$$

Buckling Equation in terms of unknown stress "f"

$$f = \frac{\eta\pi^2 E}{\left(L'/\rho\right)^2} = \frac{E_t}{E}\frac{\pi^2 E}{\left(L'/\rho\right)^2}$$

Substitute Plasticity Factor

Iterate unknown stress "f" until RHS=LHS

$$\underbrace{f}_{\text{LHS}} = \underbrace{\left[\frac{1}{1 + \frac{3}{7}n\left(\frac{f}{F_{0.7}}\right)^{n-1}}\right]\frac{\pi^2 E}{\left(L'/\rho\right)^2}}_{\text{RHS}}$$

Then → $F_{cr} = f$

Column Buckling Example: Inputs

Material

Ec	1.060E+07	psi
F0.7	64922	psi
n	19	

Cross-Section

Circular hollow tube		
D, outer dia.	1.5	in
t, wall thk.	0.06	in

Case 1

Length & B.C. coeff.

L	20	in
c	1.5	

Case 2

Length & B.C. coeff.

L	30	in
c	1.5	

Use the procedure on the previous page to compute F_{cr} for these 2 different column lengths

Column Buckling Example: Results

Material

Ec	1.060E+07	psi
F0.7	64922	psi
n	19	

Cross-Section

Circular hollow tube		
D, outer dia.	1.5	in
t, wall thk.	0.06	in

Case 1

Length & B.C. coeff.

L	20	in
c	1.5	

Verify you get these results

Fcr	57017	psi

Buckling is in Plastic Range

Case 2

Length & B.C. coeff.

L	30	in
c	1.5	

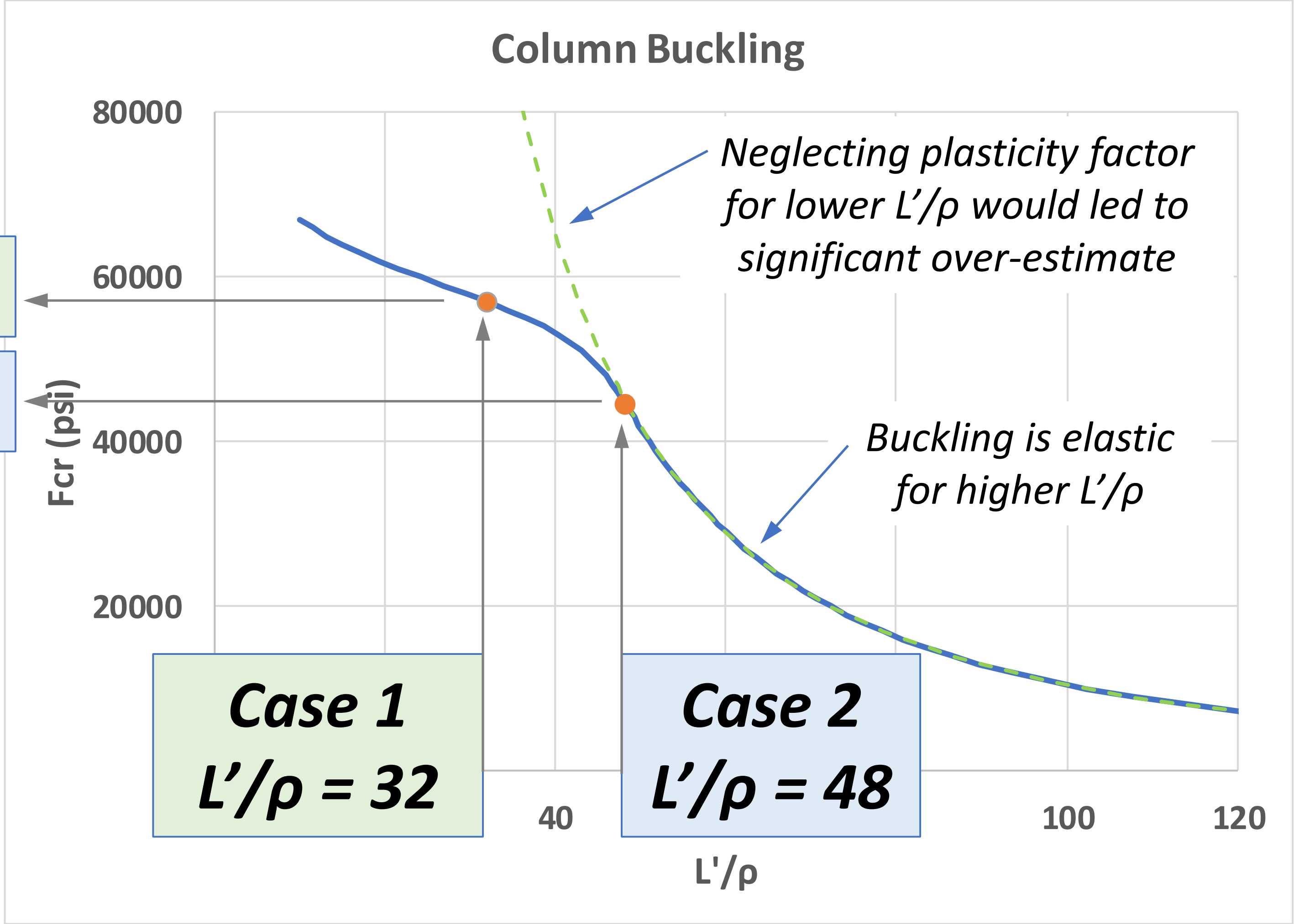
Fcr	44812	psi

Buckling is in Elastic Range

Column Buckling Example: Results

Case 1 Fcr = 57.0 ksi (plastic)

Case 2 Fcr = 44.8 ksi (elastic)



Plasticity Factors for Plate Buckling

Long Simply-Supported Plate under Compression

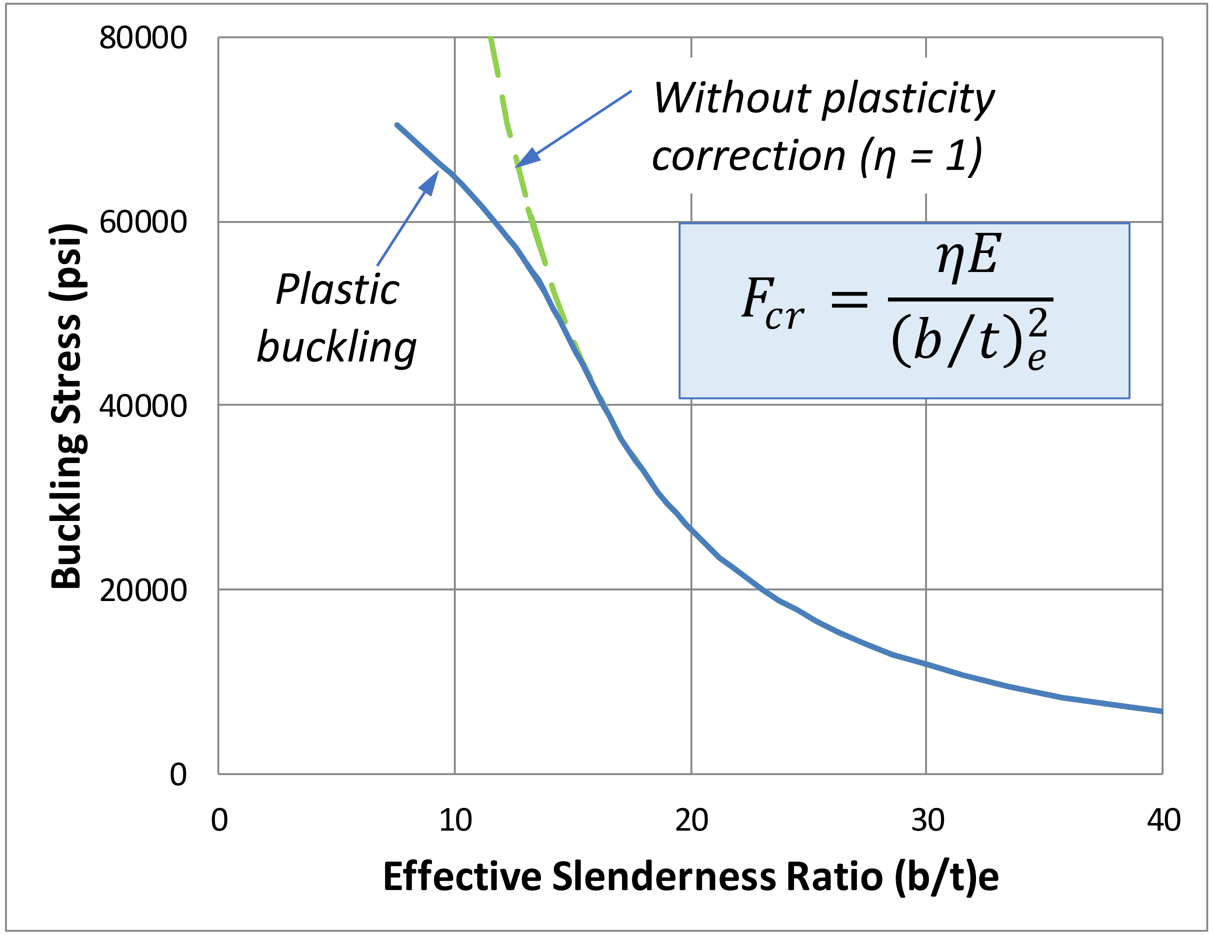
$$F_{cr} = \frac{\eta k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

For long plate under compression with simply-supported edges:

$$\eta = \frac{E_s}{E}\left\{\frac{1}{2}+\frac{1}{4}\left[1+3\frac{E_t}{E_s}\right]^{1/2}\right\}\left(\frac{1-\nu_e^2}{1-\nu^2}\right)$$

The plasticity correction factor for plates is somewhat more complicated than that for columns!

Plate Buckling



$$F_{cr} = \frac{\eta E}{(b/t)_e^2}$$

Terms in the Plate Plasticity Factor

Get E_s/E and E_t/Es from Ramberg-Osgood Equation

$$\varepsilon = \frac{f}{E} + \frac{3}{7}\frac{F_{0.7}}{E}\left(\frac{f}{F_{0.7}}\right)^n$$

$$\frac{E_s}{E} = \frac{1}{1 + \frac{3}{7}\left(\frac{f}{F_{0.7}}\right)^{n-1}}$$

$$\frac{E_t}{E_s} = \frac{1 + \frac{3}{7}\left(\frac{f}{F_{0.7}}\right)^{n-1}}{1 + \frac{3}{7}n\left(\frac{f}{F_{0.7}}\right)^{n-1}}$$

Poisson's Ratio varies in Plastic Range

$$\nu = \nu_p - \frac{E_s}{E}\left(\nu_p - \nu_e\right)$$

Usually: $\nu_p = 0.5$

Then → $\nu = 0.5 - \frac{E_s}{E}\left(0.5 - \nu_e\right)$

ν = Poisson's Ratio at given Stress
ν_e = elastic Poisson's Ratio
ν_p = fully plastic Poisson's Ratio (0.5 = incompressible)

Breakdown of the Plate Plasticity Factor

A

$$F_{cr} = \frac{\eta k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

A **B** **C** **D**

$$\eta = \frac{E_s}{E}\left\{\frac{1}{2}+\frac{1}{4}\left[1+3\frac{E_t}{E_s}\right]^{1/2}\right\}\left(\frac{1-\nu_e^2}{1-\nu^2}\right)$$

Would be too cumbersome to insert all these into one long equation, better to keep them separate and build up the calculation in steps

B
$$\frac{E_s}{E} = \frac{1}{1+\frac{3}{7}\left(\frac{f}{F_{0.7}}\right)^{n-1}}$$

C
$$\frac{E_t}{E_s} = \frac{1+\frac{3}{7}\left(\frac{f}{F_{0.7}}\right)^{n-1}}{1+\frac{3}{7}n\left(\frac{f}{F_{0.7}}\right)^{n-1}}$$

D (**B**)
$$\nu = 0.5 - \frac{E_s}{E}(0.5-\nu_e)$$

Problem Set Up

Buckling Equation in terms of unknown stress "f"

$$f = \frac{\eta k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

Plasticity factor computed as shown on previous chart

where: $\eta = function(f)$

Iterate unknown stress "f" until RHS=LHS

$$\underbrace{f}_{\text{LHS}} = \underbrace{\frac{\eta k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2}_{\text{RHS}}$$

LHS **RHS**

Because η is a nonlinear function of unknown stress "f" need to iterate

Then → $F_{cr} = f$

Plate Buckling Example: Inputs

Material

Ec	1.06E+07	psi
F0.7	65188	psi
n	15	
nu,e	0.33	
nu,p	0.5	

Thickness:
t = 0.1 in

Boundary Condition:
k = 4

Case 1

Width:
b = 4 in

Case 2

Width:
b = 2 in

Use the procedure on the previous page to compute F_{cr} for these 2 different plate widths

Plate Buckling Example: Results

Material

Ec	1.06E+07	psi
F0.7	65188	psi
n	15	
nu,e	0.33	
nu,p	0.5	

Thickness:
t = 0.1 in

Boundary Condition:
k = 4

Case 1

Width:
b = 4 in

F_{cr} ***= 24.5 ksi***
Buckling is in elastic range

Case 2

Width:
b = 2 in

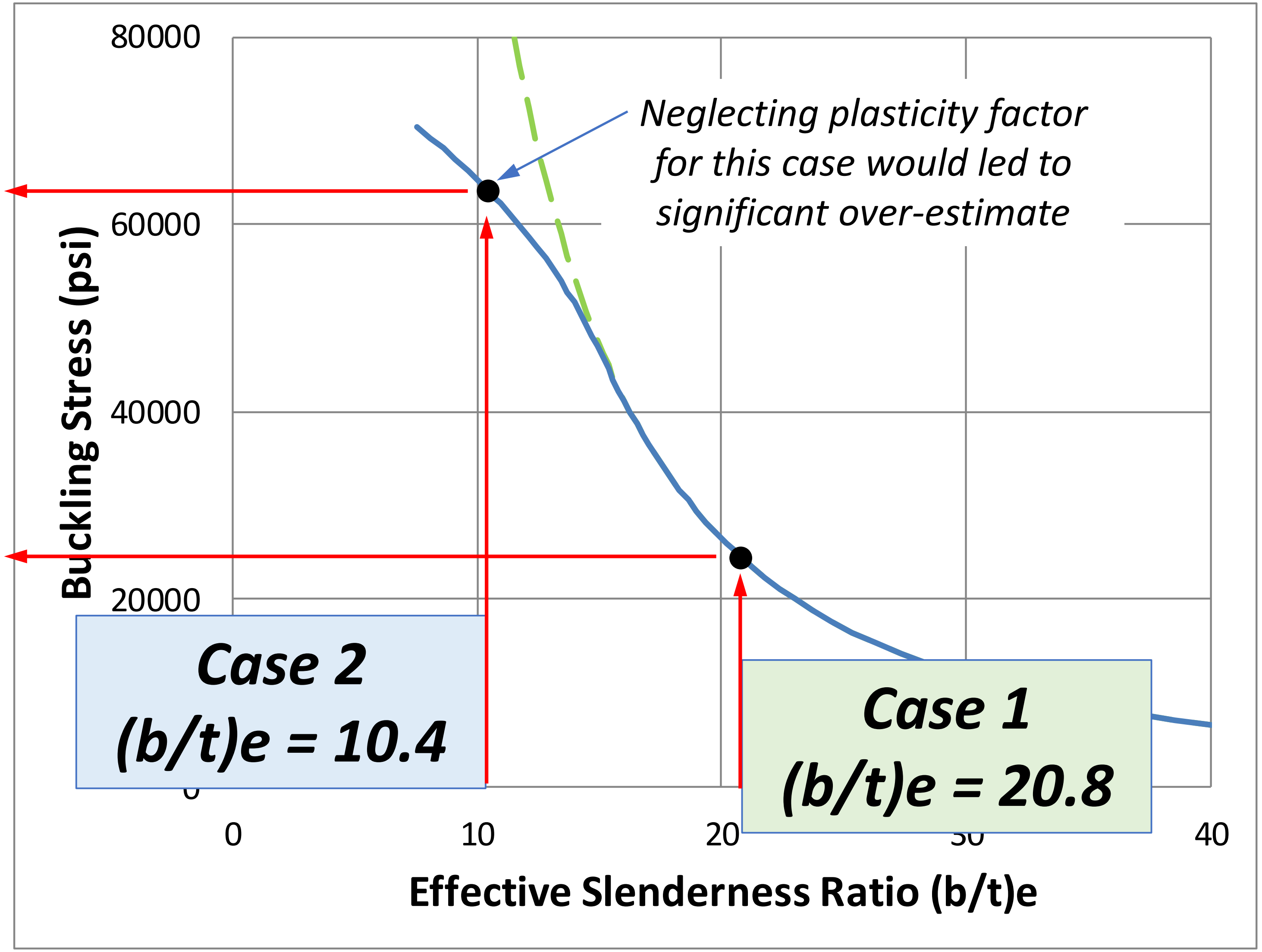
F_{cr} ***= 63.7 ksi***
Buckling is in plastic range

Verify you get these results

Plate Buckling Example: Results

Case 2 Fcr = 63.7 ksi (plastic)

Case 1 Fcr = 24.5 ksi (elastic)



Other Plate Plasticity Factors

Other Loads/Boundary Conditions

NACA TN 3781

Loading	Structure	η/j
Compression	Long flange, one unloaded edge simply supported	1
	Long flange, one unloaded edge clamped	$0.330 + 0.335\left[1 + \left(3E_t/E_s\right)\right]^{1/2}$
	Long plate, both unloaded edges simply supported	$0.500 + 0.250\left[1 + \left(3E_t/E_s\right)\right]^{1/2}$
	Long plate, both unloaded edges clamped	$0.352 + 0.324\left[1 + \left(3E_t/E_s\right)\right]^{1/2}$
	Short plate loaded as a column (L/b << 1)	$0.250\left[1 + \left(3E_t/E_s\right)\right]$
	Square plate loaded as a column (L/b = 1)	$0.114 + 0.886\left(E_t/E_s\right)$
	Long column (L/b >> 1)	E_t/E_s
Shear	Rectangular plate, all edges elastically restrained	$0.83 + 0.17\left(E_t/E_s\right)$

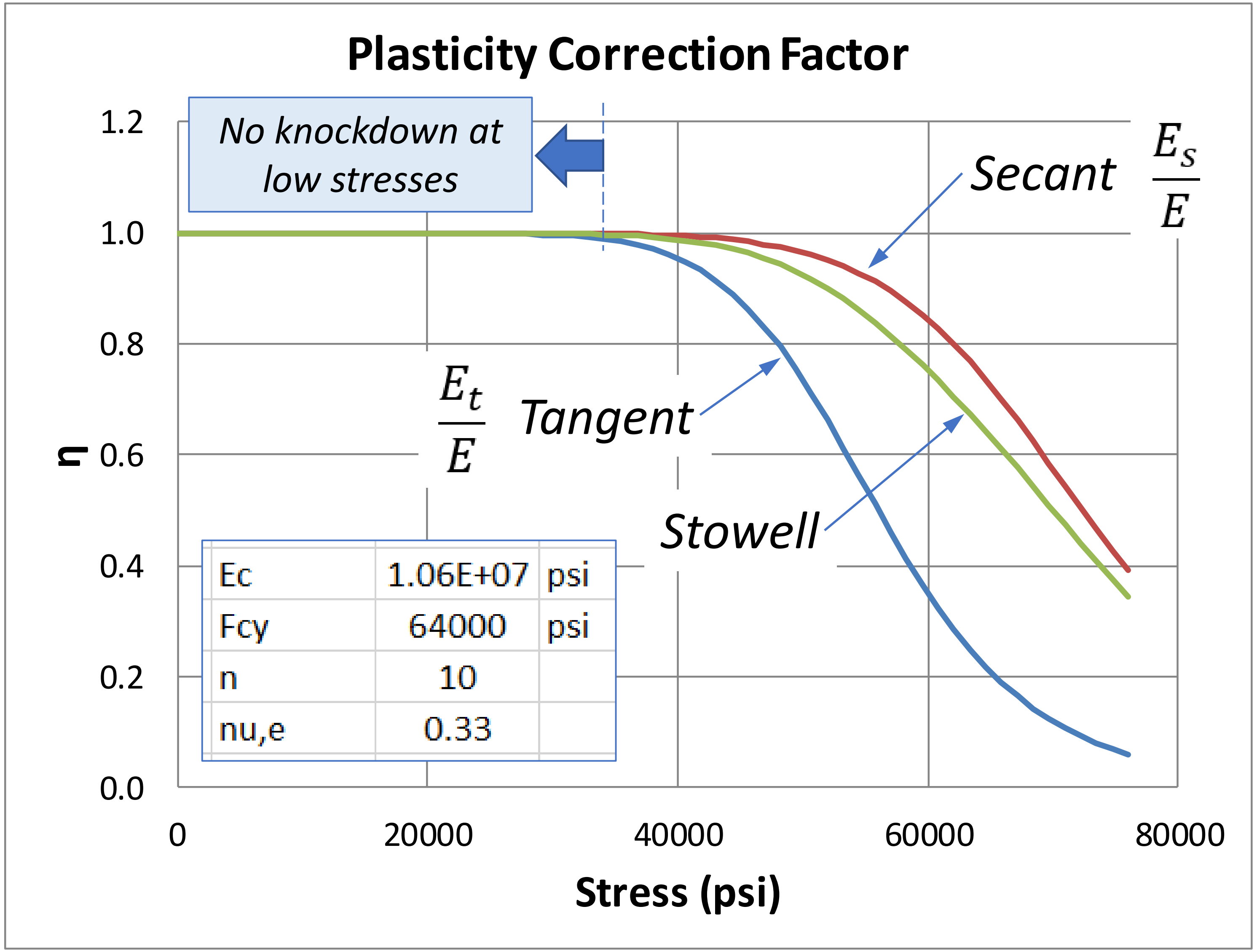
TABLE 2.- PLASTICITY-REDUCTION FACTORS

$$\left[j = \left(E_s/E\right)\left(1 - \nu_e^2\right)/\left(1 - \nu^2\right)\right]$$

This one was used for previous examples; I call it the "Stowell" Factor

Plate plasticity factors are a function of load and boundary conditions

Comparison of Plasticity Factors



The Tangent Modulus (E_t/E) gives the largest knockdown

The Secant Modulus (E_s/E) gives the smallest knockdown

The other Plasticity Factors are combinations of Tangent and Secant and therefore fall somewhere in between

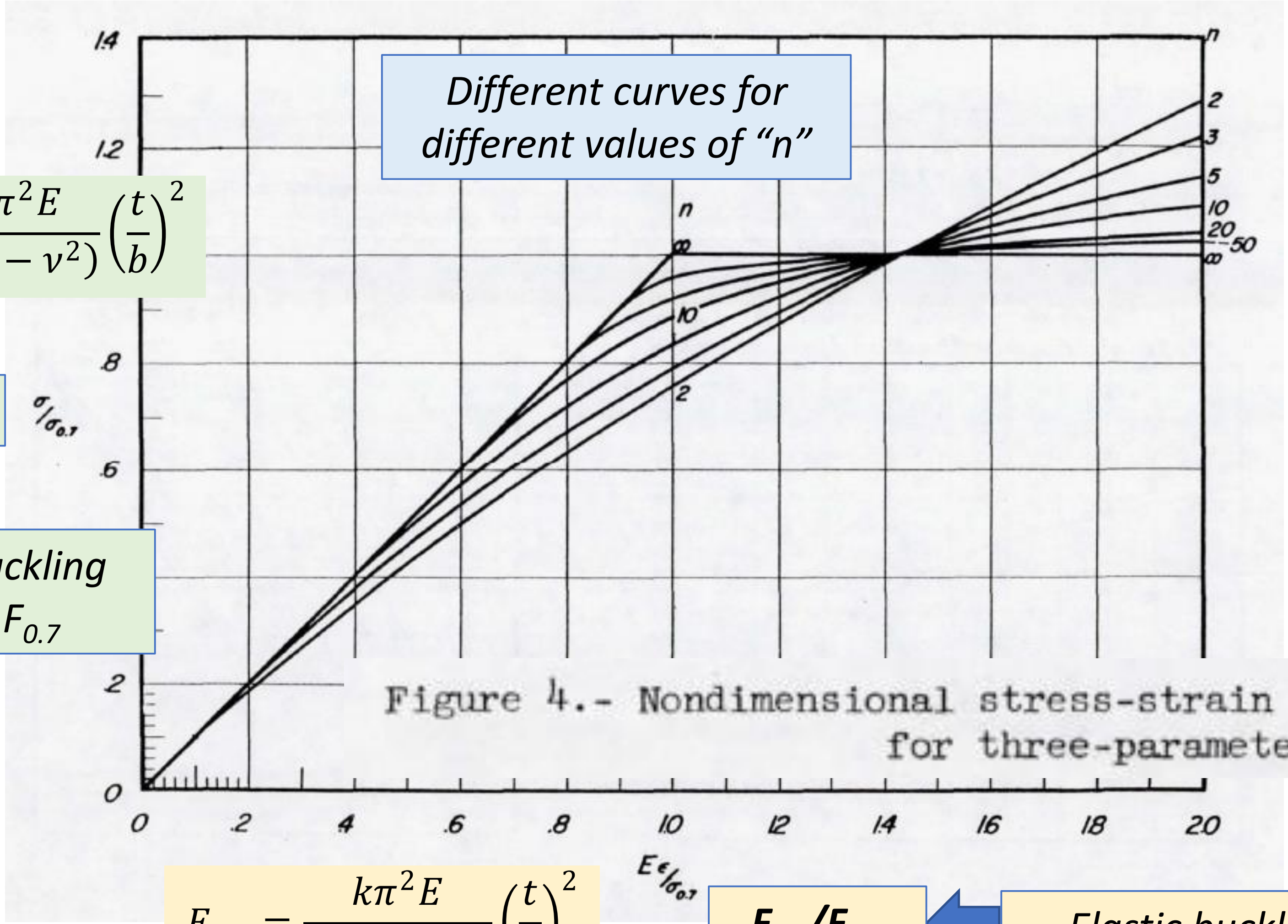
Non-dimensional Plastic Buckling Curves

NACA TN 3781

$$F_{cr} = \frac{\eta k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

$F_{cr}/F_{0.7}$

Actual buckling load ÷ $F_{0.7}$



Different curves for different values of "n"

Figure 4.- Nondimensional stress-strain curves for various values of n for three-parameter method.

$$F_{cre} = \frac{k \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

$F_{cre}/F_{0.7}$

Elastic buckling load ÷ $F_{0.7}$

FYI: Non-dimensional curves have been used as an analysis aide in the past. Nowadays, it is just as easy to compute the plasticity factor directly from the equations, as shown on the previous pages.

Summary

- Important to consider plasticity effects on buckling because:
  - Plasticity reduces the buckling load, therefore neglecting it is unconservative
  - Plasticity effect begins below the Yield Strength F_{cy}
- Plasticity effects depend on material and geometry, not applied loads
  - A thin plate (high b/t) will buckle elastically, regardless of the applied load
  - A thick plate (low b/t) will buckle plastically, regardless of the applied load
- Columns and Plates have different Plasticity Reduction Factors
  - Plates have different factors depending on loading/boundary conditions
  - Most factors are functions of E_t and E_s; not always simple substitution of E_t for E in the buckling equation
- Solution Approaches
  - Pick values and/or interpolate from previously prepared curves/tables
  - Solve the equations as needed for specific cases (e.g. using iteration)

References

1. Handbook of Structural Stability, Part I – Buckling of Flat Plates, NACA-TN-3781, Gerard & Becker, 1957
2. Description of Stress-Strain Curves by Three Parameters, NACA-TN-902, Ramberg & Osgood, 1943
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4. Metallic Materials and Elements for Aerospace Vehicle Structures, MIL-HDBK-5J, 2003
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6. Analysis and Design of Flight Vehicle Structures, Bruhn, 1973